International Journal of Theoretical Physics, Vol. 27, No. 10, 1988

Quadratic Kinetic Equations Are Linear in the Tensor Product Space

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Received March 11, 1988

It is shown that quadratic kinetic equations are linear in the tensor product space.

The technique presented below and applied to the Boltzmann equation renders quadratic operators linear and can be directly applied to equations having polynomial-type nonlinearities and describing the evolution of probability density functions or distributions.

The Boltzmann equation [adapted from Lebowitz (1987) can be written as

$$\begin{aligned}\partial/\partial t f(t, x, v_1) + v_1 \cdot \nabla_x f(t, x, v_1) + 1/m\, F(x) \cdot \nabla_v f(t, x, v_1)\\ = \int\int |v_1 - v_2| \sigma(|v_1 - v_2|, \Omega)[f(t, x, v_1')f(t, x, v_2')\\ - f(t, x, v_1)f(t, x, v_2)]\, d\Omega\, d^3 v_2 \qquad (1)\end{aligned}$$

where σ is the differential cross section, Ω is the direction of scattering, F is an external force field, and v_1 and v_2 are precollisional velocities:

$$v_1'(v_1, v_2, \Omega) = (v_1 + v_2 + |v_1 - v_2|\Omega)/2$$
$$v_2'(v_1, v_2, \Omega) = (v_1 + v_2 - |v_1 - v_2|\Omega)/2$$

The Boltzmann equation becomes linear if we express it in terms of $g = f \otimes f, f \in X$, where we can take $X = W^{1,1}(R_+;\ W^{1,1}(R^3 \times R^3))$, or $X =$ some other dense subspace of $L^\infty(R_+;\ L^1(R^3 \times R^3))$, $g \in X \otimes X$.

Define Π_6 by

$$(\Pi_6 g)(t, x, v) = \int\int g(t, x, v, 0, y, u)\, d^3 y\, d^3 u$$

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0020-7748/88/1000-1277$06.00/0

(since we are dealing with probability density functions, zero in the integrand can be replaced by any real number s). Π_6 is linear and for nonnegative functions of the form $f \otimes f$ with unit L^1 norm with respect to the last two variables, it happens to coincide with the square root in the sense of tensor product (i.e., its value on $f \otimes f$ is f).

In terms of g, the Boltzmann equation becomes linear:

$$\begin{aligned}\{\partial/\partial t\,(\Pi_6 g) + v_1 \cdot \nabla_x(\Pi_6 g) \\ + 1/m\, F(x) \cdot [\nabla_v(\Pi_6 g)]\}(t, x, v_1) = Lg)(t, x, v_1)\end{aligned} \tag{2}$$

where

$$\begin{aligned}(Lg)(t, x, v_1) = \int\int |v_1 - v_2| \sigma(|v_1 - v_2|, \Omega)[g(t, x, v_1; t, x, v_2') \\ - g(t, x, v_1; t, x, v_2)]\, d\Omega\, d^3 v_2\end{aligned}$$

[Note that Π_6 commutes with all the operators on the left side of (2).] Denoting

$$\begin{aligned}(A_0 g)(t, x, v) = (Lg)(t, x, v) - v \cdot [\nabla_x(\Pi_6 g)](t, x, v) - 1/m\, F(x) \\ \cdot [\nabla_v(\Pi_6 g)](t, x, v)\end{aligned}$$

and using the fact that Π_6 commutes with $\partial/\partial t$, we obtain the following form of the Boltzmann equation:

$$\Pi_6(\partial/\partial t\, g) = A_0 g \tag{3}$$

Because of the assumptions made about the form of g and about $f(t,\cdot,\cdot)$ we need to consider the above equation together with the constraints

$$(\Pi_6 g)(t, x, v) = (\Pi^6 g)(t, x, v) \tag{4}$$

$$({}^{t,s}\Pi_{12} g) = 1 \qquad \text{for all} \quad s \text{ and } t \tag{5}$$

$$g(t, x, v, s, y, u) \leq 0 \tag{6}$$

where

$$(\Pi^6 g)(t, x, v) = \int\int g(0, y, u, t, x, v)\, d^3 y\, d^3 u$$

$$({}^{t,s}\Pi_{12} g) = \int\int\int\int g(t, x, v, s, y, u)\, d^3 x\, d^3 v\, d^3 y\, d^3 u$$

Constraint (4) can be incorporated into the equation. Constraint (5) is the normalization condition and, since equation (2) is linear, it can be taken into account by normalizing a solution satisfying (6). Let

$$A=\begin{bmatrix} A_0 & 0 \\ 0 & \Pi^6-\Pi_6 \end{bmatrix}, \qquad B_0=\begin{bmatrix} \Pi_6\,\partial/\partial t & 0 \\ 0 & 0 \end{bmatrix}, \qquad G=\begin{bmatrix} g \\ g \end{bmatrix}$$

Then (3) and (4) give

$$B_0G=AG \tag{7}$$

which is equivalent to

$$B\,\partial/\partial t\,G=AG$$

where

$$B=\begin{bmatrix} \Pi_6 & 0 \\ 0 & 0 \end{bmatrix} \tag{18}$$

It is not necessary to take g equal to the full tensor product $f\otimes f$. Defining $g(t, x, v, y, u)=f(t, x, v)\cdot f(t, y, u)$ would suffice, but the analog of Π_6 namely the integral operator $\int\int d^3y\, d^3u$, not involving evaluation at 0) might not commute with $\partial/\partial t$ if continuity of $\partial f/\partial t$ is not uniform in position and velocity.

REFERENCES

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